Solution of the Equations of Change by Galerkin's Method

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Many of the important chemical engineering problems in the transport processes have been solved only through the use of empirical correlations relating the mass transfer coefficients, heat transfer coefficients, and friction factors to the system variables. In a few simple systems analytical solutions of the equations of change have provided verification of these correlations in the region of laminar flow. For many systems however no analytical solutions exist, and the empirical correlations which are available are not completely satisfactory. The development of high-speed computers has made it feasible to use approximate methods to obtain velocity, pressure, temperature, and concentration profiles in systems at rest or in laminar flow from the equations of change. The present investigation is concerned with the adaptation of Galerkin's method to such calculations.

Consider a transport process which is occurring within a region V surrounded by a physical or mathematical boundary surface S. For a Newtonian fluid with constant physical properties the governing differential equations (*1*) are the equation of continuity

$$(\nabla \cdot \mathbf{v}) = 0 \tag{1}$$

the equation of motion

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla \mathcal{P} + \mu \nabla^2 \mathbf{v} \tag{2}$$

the equation of energy

$$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + Q_V \tag{3}$$

and the equation of continuity of species A in a binary system:

$$\frac{Dc_A}{Dt} = \mathcal{D}_{AB} \nabla^2 c_A + R_A \tag{4}$$

These partial differential equations are of the form

$$F\left(x_i, t, Y_l^*, \frac{\partial Y_l^*}{\partial x_i}, \frac{\partial Y_l^*}{\partial t}, \frac{\partial^2 Y_l^*}{\partial x_i \partial x_j}, \cdots\right) = 0 \text{ in } V \tag{5}$$

A boundary condition of the form

$$G_S\left(Y_l^*, \frac{\partial Y_l^*}{\partial x_i}, x_i, t\right) = 0 \text{ on S for } t \geqq t_0 \tag{6}$$

will exist for each of the variables Y_l^* in any particular problem. In addition if the system considered is not at steady state, an initial condition

$$G_0(x_i, Y_l^*) = 0 \text{ in } V \text{ at } t = t_0 \tag{7}$$

must be available for each of the variables Y_l^*. It is understood that one or more unknown variables Y_l^* may appear in each of the initial and boundary conditions, as well as in the differential equations.

APPROXIMATE METHODS

Finite difference methods provide a way of obtaining approximate solutions to the above equations (*2, 3*). Many engineering problems have been analyzed successfully by this procedure. The partial differential equations are approximated by difference equations which are then solved at a set of mesh points. The accuracy obtainable by such methods is satisfactory when the mesh size is properly chosen; however the numerical calculations may be quite lengthy. The major disadvantage of such methods is that the solution is obtained only at a finite number of mesh points.

An alternative method is to assume trial functions in equation form

$$Y_l = Y_l(x_i, t, C_{jl}) \quad j = 1, \ldots, n \tag{8}$$

where the C_{jl} are adjustable parameters. An approximate solution is then obtained by determining the parameters C_{jl} so that the functions Y_l satisfy the partial differential equations and boundary conditions as well as possible. Increased accuracy can be obtained by using more parameters in the functions Y_l. Optimum values of the C_{jl} may be determined by variational calculus or by error distribution methods.

Variational calculus replaces the partial differential equation and its boundary conditions with an equivalent variational principle in which an integral must be extremalized. The proposed trial function is substituted into the integral, and the extremalization of the integral generates a set of algebraic equations which yield the optimum values of the parameters C_{jl}. In some cases a pair of reciprocal variational principles exist (*4, 5*) which provide upper and lower bounds on the solution.

Optimum values of the C_{jl} can also be determined by using one of several available error distribution principles

(2, 3). These principles distribute the error in the approximate solution over the domain of the problem in accordance with a prescribed criterion. The method of collocation requires that the error vanish at a selected set of points. The least-squares method requires that the mean-square error be made a minimum. Orthogonality methods specify that the error should be orthogonal to a chosen set of linearly independent weighting functions. Galerkin's method is a special type of orthogonality method and will be described in detail presently. More complete discussions of error distribution methods are available in Collatz (2) and Crandall (3). The error distribution methods have the advantage of working directly with the differential equations rather than with an equivalent variational problem.

GALERKIN'S METHOD

In the original form of Galerkin's method the solutions $Y_l^*(x_i, t)$ of a set of partial differential equations are represented by linear combinations of a basic set of functions Φ_j:

$$Y_l = \sum_{j=1}^{n} C_{jl}\,\Phi_j\,(x_i, t) \tag{9}$$

The Φ_j must be linearly independent, and differentiable to the extent that all terms in the differential equations and boundary conditions can be obtained (6). When the region of interest is finite, the Φ_j are commonly chosen as the n lowest-order members of a polynomial or trigonometric series expansion in the variables x_i and t. Symmetry considerations may often be used to eliminate unnecessary terms from such expansions.

The convergence of approximate solutions such as Equation (9) to the exact solution as n becomes larger merits consideration. For certain types of problems mathematical proofs of convergence are available (7); such a proof allows the determination of various sets of functions Φ_j that are complete for the particular problem and for the particular method of determining the C_{jl}. If the Φ_j are chosen as the first n members of such a set, the approximate solution must approach the exact solution as $n \to \infty$. No such convergence proof is available for the general problem statement in Equations (5), (6), and (7); however on physical grounds convergence is expected if the Φ_j are the first n members of a set of functions which is complete in the sense that as $n \to \infty$ any nontrivial function Y_l^* could be represented exactly in the region of interest. For a one-dimensional system in the region $0 < x < l$ the sets of functions

$$\Phi_0 = 1, \Phi_j = \cos\frac{j\pi x}{l} \quad (j = 1, 2 \ldots)$$

and

$$\Phi_0 = 1, \Phi_j = x^j \quad (j = 1, 2 \ldots)$$

are both complete (6) in the latter sense.

A practical test of the convergence of Galerkin's method can be made by comparing the approximate solutions obtained for successively larger values of n in Equation (9). Such calculations have shown that in many cases a small value of n gives accurate results. In such calculations it is advisable to take the trial function members Φ_j in order from a complete set.

Although any complete set of functions Φ_j may be used, it is convenient to choose an expansion which identically satisfies either the boundary conditions or the partial differential equations. This usually enables one to obtain an accurate approximate solution with fewer terms in the trial function. For instance if the trial function for each of the dependent variables Y_l identically satisfies the boundary conditions, only the interior errors

$$\epsilon_V = F(Y_l) \tag{10}$$

are nonzero. That is when the variables Y_l^* are replaced by the trial functions Y_l in any of Equations (5), the corresponding function F may deviate from zero by an amount ϵ_V. This error is distributed by making it orthogonal to the functions Φ_j throughout the region V. The orthogonality relations

$$\int_t \int \int_V \int F\,(Y_l)\,\Phi_j\,dV dt = 0 \quad (j = 1, 2, \ldots n) \tag{11}$$

provide algebraic equations in the C_{jl}. Such a method of solution is called an *interior method.*

On the other hand the trial functions may be chosen so that the differential equations are identically solved. Such a method of solution is known as a *boundary method* since the only sources of error are the boundary errors

$$\epsilon_S = G_S(Y_l) \tag{12}$$

which represent the deviations of the approximate solutions from the boundary conditions on S, and the initial errors

$$\epsilon_0 = G_0(Y_l) \tag{13}$$

which represent the errors in the approximate solutions at time t_0. These errors are then made orthogonal to the Φ_j over the appropriate regions. The resulting orthogonality relations

$$\int_t \int_S \int G_S\,(Y_l)\,\Phi_j\,dS\mathrm{d}t = 0 \quad j = 1, 2, \ldots n \tag{14}$$

and

$$\int \int_V \int G_0\,(Y_l)\,\Phi_j\,dV = 0 \text{ at } t = 0, \quad j = 1, 2, \ldots n \tag{15}$$

yield algebraic equations for the C_{jl}.

In some problems it is not feasible to choose a set of trial functions which satisfies either the differential equations or the boundary conditions. For such systems all three types of orthogonality relations [Equations (11), (14), and (15)] are needed, and the calculation is referred to as a *mixed method.*

For problems connected with variational principles Galerkin's method is closely related to the Ritz variational method. In a number of instances Galerkin's method is equivalent to the Ritz variational method in that it leads to the same approximate solution, usually with simpler computations (6). The authors believe that in such cases Galerkin's method is superior, since it allows closer contact with the actual physical problem.

The other error distribution principles are very similar to Galerkin's method. Usually the convergence of Galerkin's method with increasing n is more rapid than that of collocation or least squares (3, 8). By using trial functions which satisfy the boundary conditions and have the proper symmetry properties accurate results can often be obtained with very small values of n.

APPLICATION OF GALERKIN'S METHOD TO TRANSPORT PROBLEMS

The Galerkin method is readily adapted to transport problems. The interior orthogonality relations [Equation (11)] are written for the equation of continuity

$$\int_t \int \int_V \int (\nabla \cdot \mathbf{v})\,\Phi_j\,(x_i, t)\,dV\,dt = 0 \tag{16}$$

the equation of motion

$$\int_t \int \int_V \int \left(\rho\frac{D\mathbf{v}}{Dt} + \nabla\mathcal{P} - \mu\,\nabla^2\mathbf{v}\right)\Phi_j(x_i, t)\,dV dt = 0 \tag{17}$$

the equation of energy

$$\int_t \int \int_V \int \left(\rho \hat{C}_p \frac{DT}{Dt} - k \nabla^2 T - Q_V \right) \Phi_j(x_i, t)\, dV\, dt = 0 \quad (18)$$

and the equation of continuity of A

$$\int_t \int \int_V \int \left(\frac{Dc_A}{Dt} - \mathcal{D}_{AB} \nabla^2 c_A - R_A \right) \Phi_j(x_i, t)\, dV\, dt = 0 \quad (19)$$

In addition the boundary conditions on each of the dependent variables Y_l will yield a set of orthogonality relations on the surface [Equation (14)], and the initial conditions will produce a set of orthogonality relations at $t = t_0$ [Equation (15)]. The use of this method can be considered to be a three step procedure: a suitable trial function for each of the independent variables must be selected, these trial functions must be substituted into the appropriate sets of Equations (14), (15), (16), (17), (18), and (19), and the resulting equations in the parameters C_{jl} must be solved to obtain the approximate solutions in terms of the chosen trial functions.

The selection of trial functions can be simplified by allowing the coefficients C_{jl} to depend on one of the independent variables. For example consider the trial function

$$Y_l = \sum_{j=1}^{n} C_{jl}(t)\, \phi_j(x_i) \quad (20)$$

With trial functions of this type only the position dependence is directly assumed. Equations (14), (15), (16), (17), (18), and (19) can then be modified by substituting $\phi_j(x_i)$ for $\Phi_j(x_i, t)$ and deleting the integration over time. For instance Equation (17) becomes

$$\int \int_V \int \left(\rho \frac{D\mathbf{v}}{Dt} + \nabla \mathcal{P} - \mu \nabla^2 \mathbf{v} \right) \phi_j(x_i) dV = 0 \quad (21)$$

Evaluation of these integrals yields ordinary differential equations in the C_{jl}. In steady state problems the C_{jl} do not depend on time, and the equations in the C_{jl} are algebraic.

Solutions based on Equation (20) can resolve the time-dependent behavior more completely, since the errors can be orthogonalized at every instant of time instead of being orthogonalized over a finite period of time. Also in most problems a smaller set of functions may be used, because the ϕ_j do not depend on t. This procedure has been tested on a number of transient heat conduction problems, with a complete set of functions $\phi_j(x_i)$ used, and the results have agreed in each case with the classical method of separation of variables. The set of functions $\phi_j(x_i)$ should be chosen to satisfy the boundary conditions, differential equations, and symmetry characteristics of the problem to the extent that this is feasible.

When the trial functions satisfy the boundary conditions, there is usually no difficulty in substituting the Y_l into Equations (16), (17), (18), and (19) to obtain a determinate set of equations for the parameters. However when a mixed method is used, Equations (16), (17), (18), and (19) plus the boundary conditions as expressed in Equation (14) and (15) will yield more equations than there are unknowns. Therefore some of the equations must be discarded to avoid an overdetermined system. The choice of equations in this case is somewhat arbitrary as long as the equations are independent; consideration should be given to the characteristics of the solution which are most important in a particular problem. It may be best to simply discard the requirement that the errors ϵ_V be orthogonal to the highest-order weighting functions. Further study of this aspect of mixed methods would be useful.

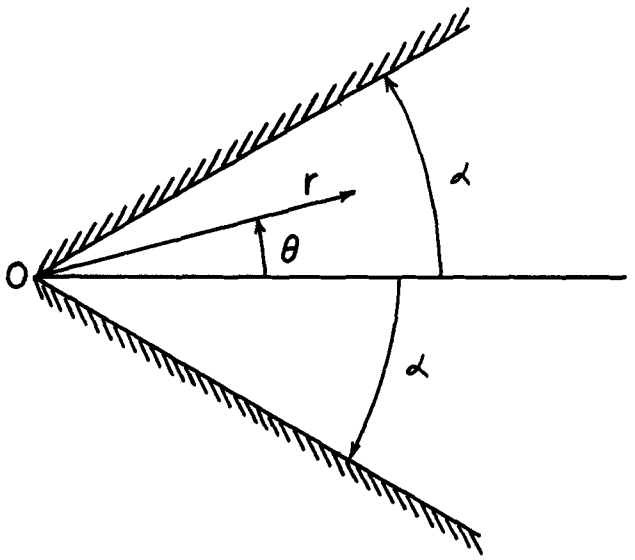

Fig. 1. Coordinate system for flow between inclined planes.

The calculation of velocity profiles is the most difficult of the problems; therefore the equations for the C_{jl} of a velocity trial function merit discussion. The equation of continuity yields linear algebraic equations in the C_{jl}. If the term $[\mathbf{v} \bullet \nabla \mathbf{v}]$ is zero or negligible, the equation of motion also provides linear equations in the C_{jl} (ordinary differential equations for time-dependent flows or algebraic equations for steady state flows). In either case the equations are relatively easy to solve. If on the other hand $[\mathbf{v} \bullet \nabla \mathbf{v}]$ is nonzero, the equations are quadratic in the C_{jl}. The solution of a set of quadratic equations is more difficult, but it is feasible with modern computing techniques. It is possible in such nonlinear problems that several different solutions for the C_{jl} exist. In such cases the physically realistic solutions are chosen from these mathematically possible solutions. A specific example of this procedure will be discussed later.

To demonstrate the procedure outlined above, Galerkin's method has been applied to a nonlinear fluid flow problem. The convergence of the approximate solution is also demonstrated by comparison with a known exact solution.

EXAMPLE: STEADY STATE FLOW BETWEEN TWO INCLINED PLANE WALLS

The steady flow of a Newtonian fluid with constant ρ and μ between two inclined plane walls has been investigated previously (*9, 10*) and provides a good test of Galerkin's method. The two planes meet at an angle 2α as shown in Figure 1, so that flows in the $+r$ direction are divergent and flows in the $-r$ direction are convergent. Cylindrical coordinates are used with the z axis perpendicular to the cross section shown in Figure 1 at point 0. The flow is assumed to be radial ($v_\theta = 0$, $v_z = 0$) and independent of z [$v_r = v_r(r, \theta)$]. The mass outflow across any surface of constant r, per unit time and unit width in the z direction, is Q. A Reynolds number $N_{Re} = Q/\mu$ is defined for this system so that N_{Re} is positive for flows in the $+r$ direction and negative for flows in the $-r$ direction. The differential equations describing this system are the r component of the equation of motion

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial \mathcal{P}}{\partial r}$$

$$+\mu\left[\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r)\right)\right] \quad (22)$$

the θ component of the equation of motion

$$-\frac{1}{r}\frac{\partial \mathcal{P}}{\partial\theta}+\frac{2\mu}{r^2}\frac{\partial v_r}{\partial\theta}=0 \quad (23)$$

and the equation of continuity

$$\frac{\partial}{\partial r}(r\,v_r)=0 \quad (24)$$

On the walls of the system the velocity must be zero; therefore

$$v_r=0 \text{ at } \theta=\pm\alpha \quad (25)$$

In addition the condition of a specified total mass flow rate

$$2\rho\int_0^{\alpha} v_r\, r\, d\theta = Q \quad (26)$$

has been used here as the remaining boundary condition.

The exact solution for this problem is available in terms of elliptic integrals (9) or in terms of elliptic functions (10). To facilitate comparisons the notation of (9) will be used; it is shown there that the condition of purely radial flow gives

$$v_r=\frac{6\mu}{\rho r}u(\theta) \quad (27)$$

and

$$\mathcal{P}=\frac{6\mu^2}{\rho r^2}[2\,u(\theta)-K_{\mathrm{I}}]+K_{\mathrm{II}} \quad (28)$$

where u is a function of θ and K_{I}, K_{II} are constants.

In the present analysis Galerkin's method was used to obtain an approximate solution for $u(\theta)$. A number of function sets $\phi_j(\theta)$ which satisfied the boundary conditions at the wall were initially considered. The functions which yielded the best approximate solution with only the three leading terms were then used in solving the problem. These trial functions are

$$v_r=\sum_{j=1}^{n}\left(\frac{Q}{\rho r}\right)C_{jv}\,(\alpha^2-\theta^2)^j+\left(\frac{Q}{\rho r}\right)C_{0v} \quad (29)$$

and

$$\mathcal{P}-\mathcal{P}_0=\sum_{j=1}^{n}\left(\frac{Q\mu}{\rho r^2}\right)C_{j\mathcal{P}}\,(\alpha^2-\theta^2)^j+\left(\frac{Q\mu}{\rho r^2}\right)C_{0\mathcal{P}} \quad (30)$$

With these trial functions the equation of continuity is identically satisfied; the θ component of the equation of motion is satisfied by setting $C_{j\mathcal{P}}=2\,C_{jv}$ for $j=1,\ldots,n$, and the boundary conditions at the wall are satisfied by setting $C_{0v}=0$. The boundary condition on the flow rate [Equation (26)] becomes

$$2\int_0^{\alpha}\left(\sum_{j=1}^{n}C_{jv}\,(\alpha^2-\theta^2)^j\right)d\theta-1=0 \quad (31)$$

When the trial functions are substituted into the r component of the equation of motion and ϵv is orthogonalized as in Equation (21), the following set of equations is obtained:

$$\int_0^{\alpha}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}N_{Re}\,C_{iv}\,C_{jv}\,(\alpha^2-\theta^2)^{i+j}+2\,C_{0\mathcal{P}}+4\sum_{j=1}^{n}C_{jv}\,(\alpha^2-\theta^2)^j+\sum_{j=1}^{n}C_{jv}\frac{\partial^2}{\partial\theta^2}(\alpha^2-\theta^2)^j\right](\alpha^2-\theta^2)^k\,d\theta=0 \quad (32)$$

$k=1,\ldots n$

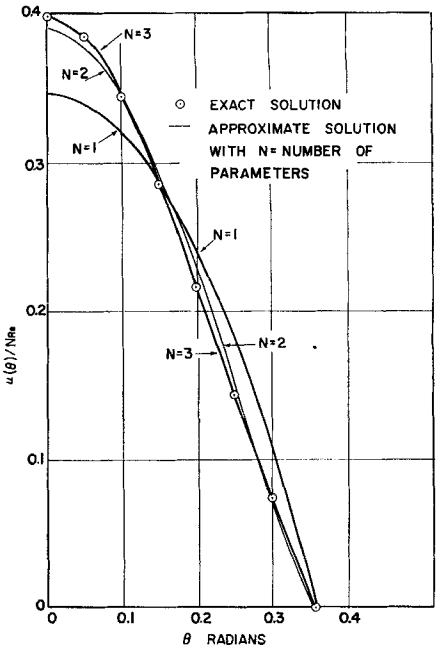

Fig. 2. Comparison of approximate and exact velocity profiles for $N_{Re}=14.164$ and $\alpha=0.360$.

An inspection of the integrals shows that the parameters C_{jv} and $C_{0\mathcal{P}}$ depend on both N_{Re} and α.

The integrals in Equations (31) and (32) were evaluated analytically to obtain one linear Equation (31) and n quadratic Equations (32) for the $n+1$ unknowns C_{jv} and $C_{0\mathcal{P}}$. This set of equations was solved on a CDC-1604 computer, with a nonlinear least-squares program similar to that available from American Institute of Chemical Engineers (11). The left sides of Equations (31) and (32) were evaluated, with numerical estimates of the parameters C_{jv} and $C_{0\mathcal{P}}$, to define initial deviations for the least-squares program; the program then minimized the sum of squares of the deviations by iterative adjustment of the parameters. A correct solution gives zero as the sum of squares of deviations thus defined, and this condition was achieved within roundoff error in all solutions reported here. A few calculations, with N_{Re} and α in or near the separated flow region, failed to converge to a zero sum of squares and are not reported.

Since the equations are nonlinear, a number of solutions for the trial function parameters may be obtained at any given values of α and N_{Re}. The determination of the physically realistic solution was made in the following manner. For a two-parameter velocity trial function two sets of values of C_{1v} and C_{2v} were obtained. The set of values which predicted the correct direction of the center-line velocity was selected as the proper solution. When a third velocity parameter was added, several solutions for C_{1v}, C_{2v}, and C_{3v} could be obtained. The set of values in which the leading parameters C_{1v} and C_{2v} were nearest to the correct two parameter solution was chosen as the correct three parameter solution. Thus the n parameter solution was obtained as a correction of the $(n-1)$ parameter solution.

An alternate approach which may be used to obtain the correct solution is to consider nonlinear flows as perturbations of the creeping flow solution. Since a unique analytic solution is available for creeping flow, velocity profiles at higher flow rates could be obtained as corrections to the creeping flow profile. Additional investigation of this technique is needed before its worth can be evaluated.

Velocity profiles were determined at three values of N_{Re} and α for which exact solutions are available (*10*). The approximate solution in the form

$$\frac{u(\theta)}{N_{Re}} = \frac{rv_r\rho}{6\mu N_{Re}} \sum_{j=1}^{n} \frac{1}{6} C_{iv} (\alpha^2 - \theta^2)^j \qquad (33)$$

is compared with the exact solution for $N_{Re} = 14.164$, $\alpha = 0.360$ in Figure 2. Other comparisons of the approximate solutions with exact solutions gave comparable agreement.

Approximate velocity profiles with as few as three parameters were found to be in good agreement with exact solutions. The addition of terms beyond C_{4v} in the trial functions had a negligible effect on the approximate solutions in regions of nonseparated flow. The rapid convergence to a satisfactory approximate velocity profile is not considered surprising, since the trial functions satisfy the boundary conditions at the walls plus two of the differential equations of this system and exhibit the known symmetry characteristics of the problem.

Velocity profiles were obtained for $\alpha = 0.5236$ (30 deg.) at several Reynolds numbers. A few of these profiles are shown in Figure 3. For large negative Reynolds numbers (which correspond to high rates of flow between converging walls) the velocity profiles become very blunt. Neither Rosenhead nor the present investigators predict separation in these types of flows. The profile for $N_{Re} = 0$ was obtained numerically as the limiting profile for very small Reynolds numbers. This profile was the same to four significant figures as the analytical solution of the equation of motion with the inertial terms neglected. As the Reynolds number increases, the profiles become less blunt, and eventually a separated flow is predicted. The onset of separation was found to occur in the range $17 < N_{Re} < 25$; the limit found analytically (*10*) is $N_{Re} = 21$. The approximate solutions for separated flows did not approach any apparent limit as the fourth and fifth terms were added to the trial functions. It appears that several more terms are needed to describe these separated flows.

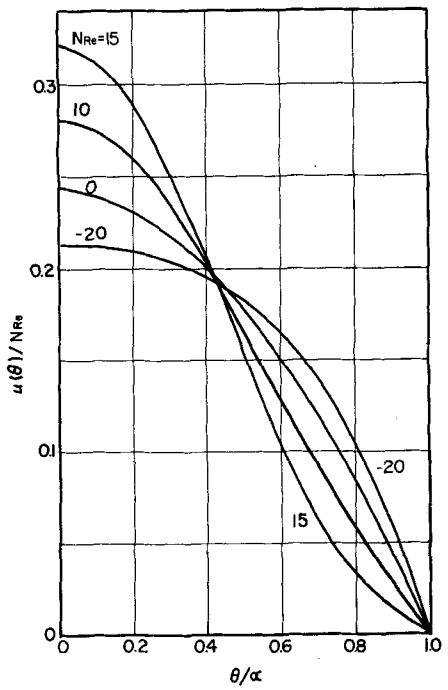

Fig. 3. Effect of Reynolds number on the velocity profile for $\alpha = 0.5236$.

CONCLUSION

An adaptation of Galerkin's method to the equations of change has been given. This method provides a straightforward procedure for obtaining an approximate solution from trial functions. Some of the advantages of Galerkin's method over other approximate methods are:

1. The solution is in the form of functions.
2. Closer contact with the physical problem is possible with Galerkin's method than with variational methods.
3. Greater flexibility in trial functions is permissible.
4. Some control over the relative accuracy of the solution at various points in the region is available with the use of interior, exterior, or mixed methods.

The problem of flow between inclined plane walls was solved by the proposed method. Application of Galerkin's method to this system proved to be much simpler than earlier attempts to solve the problem with variational methods. Accurate approximate solutions were obtained for nonseparated flows. Additional study of separated flows will be necessary before the applicability of Galerkin's method in such systems can be definitely evaluated.

An investigation of transport processes in a regular packed bed of spheres is now being made with Galerkin's method. The results of this study will be presented in a future paper.

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NOTATION

c_A = concentration of species A
C_{jv} = velocity trial function parameters in Example
C_{jl} = trial function parameters for the lth dependent variable
$\hat{C}_p$ = heat capacity
C_{jP} = pressure trial function parameters in Example
$\mathcal{D}_{AB}$ = diffusivity
F = function which expresses an equation of change in the form $F = 0$
G_S = function which expresses a surface boundary condition in the form $G_S = 0$
G_0 = function which expresses an initial condition in the form $G_0 = 0$
k = thermal conductivity
K_I, K_{II} = constants in Equation (28)
$\mathcal{P}$ = static pressure referred to a constant elevation $\mathcal{P} = p + \rho gh$
Q = mass flow rate per unit width between two inclined planes ($ML^{-1}t^{-1}$)
Q_V = volumetric rate of heat generation within the volume V
N_{Re} = Reynolds number defined as Q/μ
R_A = volumetric rate of production of species A by homogeneous chemical reaction
S = boundary surface of the region of interest
t = time
T = temperature
$u(\theta)$ = function of θ in Equation (33)

V = volume containing the region of interest in a problem
$\mathbf{v}$ = velocity vector
v_i = velocity component in the direction of the coordinate x_i
x_i = coordinates
Y_i^* = exact solution for one of the dependent variables $\mathcal{P}$, v_i, T, c_A
Y_i = approximate solution for one of the dependent variables

Greek Letters

α = angle of inclination of the plane walls
ϵ_0 = the initial error, the amount by which an approximate solution fails to satisfy the initial conditions
ϵ_V = the interior error, the amount by which an approximate solution fails to satisfy the differential equations
ϵ_S = the boundary error, the amount by which an approximate solution fails to satisfy the boundary conditions on S
μ = viscosity
ρ = density
ϕ_j = the jth member of a set of position dependent trial functions
Φ_j = the jth member of a set of general trial functions

Subscripts

A, B = chemical species in a binary system
i, j, k = summation indices
n = upper limit on summation indices
P = pressure function
S = on the surface
V = within the volume
v = velocity
0 = initial state

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Turbulent Film Condensation

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Nusselt's laminar film theory of condensation (*10*) is simple and capable of predicting the heat transfer coefficient in many cases. However the actual heat transfer coefficients are found to be somewhat higher for fluids with moderate and high Prandtl numbers, while the heat transfer coefficients observed for the liquid metals (small Prandtl number) are considerably lower than Nusselt's prediction. These lead one to investigate the effect of turbulent transports in condensation because the thickness of condensate film can become large in reality.

In the past few attempts have been made in this direction, notably by Seban (*12*), Roshenow et al. (*11*), and Dukler (*3*). Seban adopted the well-known universal logarithmic velocity distributions for a velocity profile in the condensate film. With an additional assumption of eddy thermal diffusivity being equal to eddy kinematic viscosity the heat transfer analysis was carried out by an approximate analytical method. The theory was extended by Rohsenow et al. to include the positive interfacial shear stress. In view of a recent work by Lee (*6*) the use of logarithmic velocity distribution seems to be valid only in the case of a large positive interfacial shear stress. Recently Dukler presented a more ambitious approach of actually solving the velocity profile using the eddy viscosity expressions of Deissler and von Karman. However his velocity profile suffers from the physical inconsistencies, such as the discontinuity at the intersection of two regions and the failure to satisfy the interfacial shear stress correctly. Furthermore his heat transfer results for small Prandtl number range do not seem physically plausible owing to neglecting the molecular thermal conductivity with respect to eddy conductivity.

The purpose of this paper is to recompute the heat transfer coefficients for turbulent Nusselt's model without introducing a priori assumption on the magnitude of transport coefficients. The method of solution is that developed by the author (*6*), in which a smooth velocity profile with correct interfacial shear stress is obtained. In contrast to Dukler's work the correct heat transfer coefficients are still much larger than the liquid metals data of Misra and Bonilla (*9*). This is not unexpected because the Nusselt's model does not account for the following considerations: nonvanishing interfacial shear stress, inertia effects, convective heat transfer and subcooling, interfacial waves and